PMM U.S.S.R.,Vol.52,No.2,pp.267-270,1988
Printed in Great Britain

0021-8928/88 $10.00+0.00

USE OF THE ROBEN PRINCIPLE IN SOLVING THREE-DIMENSIONAL PROBLEMS OF THE THEORY OF ELASTICITY*

P.I. PERLIN

The Roben principle is used in solving the integral equations (IE) corresponding to the fundamental spatial problems of the theory of elasticity by the method of successive approximations. It has been established that a possible divergence of the process of successive approximations caused by the lack of accuracy of the computational schemes does not lead to the solution diverging in the stresses. It has been shown that applying the method of successive approximations to the IE of the second interior problem when the boundary conditions are not selfequilibrated, yields a solution which converges in terms of the stresses, corresponding to specific selfequilibrated boundary conditions. A method of solving the boundary value problems is proposed in which the corresponding IE are situated in the spectrum and conditions for their solvability are not satisfied.

Let us consider the Fredholm IE of the second kind

$$\varphi(x) - \lambda \int k(x, y)\, \varphi(y)\, dy = f(x). \tag{1}$$

Let $\lambda = 1$ be the eigenvalue smallest in modulus. We shall solve the IE (1) using the method of successive approximations. To do this, we shall write the function $\varphi(x)$ in the form of a series

$$\varphi(x) = \sum_{n=0}^{\infty} \lambda^n \varphi_n(x) \tag{2}$$

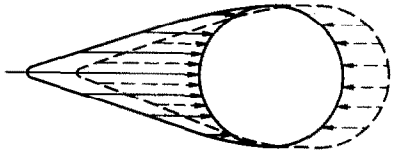

Fig.1

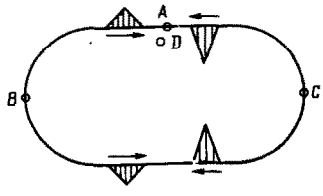

Fig.2

Then we arrive at the following recurrence relations:

$$\varphi_n(x) = \int k(x, y)\, \varphi_{n-1}(y)\, dy, \quad n = 1, 2, \ldots; \quad \varphi_0(x) = f(x). \tag{3}$$

The Roben principle /1/ consists of the assertion that constants $\tau < 1$, a exist and an integer N such that when $n > N$ and $m > n$,

$$|\varphi_n(x) - \varphi_m(x)| < a\tau^n. \tag{4}$$

From (4) it follows that $\varphi_n(x)$ tends uniformly to the limit as $n \to \infty$, and the limit (we denote it by $\varphi_1^*(x)$) is an eigenfunction of IE (1) at the point $\lambda = 1$.

If the right-hand side of (1) is orthogonal to the eigenfunctions of the allied equation, then the limit in question will be equal to zero and series (2) will lead to construction of the solution of IE.

The same arguments hold when the eigenvalue smallest in modulus is $\lambda = -1$. In this case we consider the limit

$$\lim (-1)^n \varphi_n(x) = \varphi_{-1}^*(x) \tag{5}$$

and $\varphi_{-1}^*(x)$ is an eigenfunction at the point $\lambda = -1$.

Let us express the above assertions differently. Let N be the number of terms retained in (2). Then we obtain the following expression for the sum:

$$\varphi^N(x) = \sum_{n=0}^{N} \lambda^n \varphi_n(x) = \Phi(x) + \varepsilon^N(x) + N\varphi^*(x) \tag{6}$$

*Prikl.Matem.Mekhan.,52,2,337-341,1988

where $\Phi(x)$ is a function, $\varepsilon^N(x) \to 0$ as $N \to \infty$, $\varphi^*(x)$ is an eigenfunction when $\lambda = 1$ or -1, found as the result of the passage to the limit.

Thus from (6) it follows that violating the conditions of orthogonality of the right-hand side of the IE by the eigenfunctions of the allied equation, leads to the divergence of the consecutive approximations.

We have said above that the Fredholm IE of the second kind are under consideration. It therefore seems natural that we should also consider other IE (e.g. the singular IE), provided that they have the same spectral properties. Amongst such IE we have those corresponding to the first and second boundary value problems of the theory of elasticity /2/.

In problems with a physical content, where the IE is obtained from the representation of the function sought in the form of one or other potential, it is best to consider the IE situated within the spectrum also when the conditions of their solvability are violated. The problem is that at times the solution of the integral equation itself is of no interest, and we must determine some functional. For example, in solving the second interior problem of the theory of elasticity we need to find the components of the stress tensor, while the term $\varphi^*(x)$ in (6) corresponds to the displacement of the body as a rigid entity and does not affect the stresses. Thus irrespective of the fact that the consecutive approximations diverge, the algorithm may lead to constructing a solution of the boundary value problem corresponding, naturally, to boundary conditions different from the initial conditions and obtained by applying, in a special manner, a distributed load determined by the principal vector and the moment of external forces.

In order to illustrate what has just been said, we shall consider the axisymmetric problem of the theory of elasticity for a sphere in the case when the load is applied to the left-hand hemisphere and has a single component (along the axis of symmetry)

$$\sigma = 1 - \cos\theta \quad (-\pi/2 \leqslant \theta \leqslant 0)$$

(Fig.1). The integral equation of the second fundamental problem /2/ was solved using the regular representations of singular integrals /3/. Let us compare the sums (6) obtained using 20 and 40 terms and the corresponding values of the stresses. At one of the points of the boundary surface the sum differ from each other by a factor of two, and the stresses (at one of the internal points) by 4%. The dashed line shows the envelope of the boundary stresses corresponding to the equation obtained. The stresses are calculated by subtracting from the initial boundary conditions the stresses equal to the principal vector of applied forces divided by the total surface area.

A study of such problems (where the conditions of solvability of the initial problem and hence of the corresponding integral equation, are violated), may be useful. Let us assume that the solution of the mixed problem is obtained by expanding in a series the load applied to the part of the boundary at which the displacements are specified. The method given no longer makes it necessary to construct the selfequilibrated harmonics, and this can considerably simplify the execution of the algorithm. The result of imposing the conditions on the principal vector and vector-moment will be that the additional terms will cancel during the summation; therefore the boundary conditions in terms of the stresses will be satisfied.

Since, as was said in /3/, the computational schemes are inaccurate, even in the case when the conditions of solvability of the IE are satisfied (when the corresponding integrals are exactly equal to zero), the consecutive approximations may diverge. We know that every function $\varphi_n(x)$ must be orthogonal to the eigenfunction of the allied equation. The error present in the computational scheme can lead to violation of these conditions even at the first stage, and it will get larger as the number of iterations increases. Exceptions are problems possessing one or other form of symmetry. For example, in the problems of the theory of elasticity, the presence of three planes of geometrical and force symmetry is sufficient (naturally, when the symmetry is apparent in the discretization of the surface).

In order to ensure the convergence of the iterative process, we proposed in /3/ that every iteration step be corrected as follows:

$$\varphi_n{}^k(x) = \varphi_n(x) - \varphi^*(x) \int \varphi_n(y)\, \varphi^*(y)\, dy . \qquad (7)$$

Here $\varphi^*(x)$ is the eigenfunction of the allied equation and the expression given contains only a single eigenfunction for simplicity.

In the general case we must carry out the summation over all eigenfunctions of the allied equation. In the case of the second fundamental problem of the theory of elasticity we should introduce six terms, and this becomes fairly bulky. In problems with axial symmetry only a single term is retained.

We shall treat the inaccuracy of the computational scheme as an error introduced originally into the boundary conditions, and then realize the algorithm accurately. Then, from formula (6) it follows that the process diverges (since the boundary conditions in their altered form will not satisfy the conditions of solvability of the IE). However, for the problems in question (when the Roben principle is applicable), it is found possible to obtain a converging solution for the quantities of interest from the divergent soltuions of the IE.

We have solved the axisymmetric problem of the theory of elasticity for a cylinder with rounded ends (Fig.2), with tangential loads applied to cylindrical strips. The type of load chosen was such that a change in the discretization of the computational scheme near the points at which tangential stresses were largest, cause a different degree of non-equilibration of the external loads. A coarse (1°), intermediate (2°) and fine (3°) discretization was employed. The table shows the results of computing the tangential component of the term $\varphi_n(x)$ at the 15-th iteration, of the sum with 15 terms, and of the stresses at some internal point D after 8 and 15 iterations. The brackets contain the corresponding results from the computations using Eq.(7).

The data given show that when the iterative process is realized directly the solution of IE definitely diverges, but nevertheless leads to converging values of the stresses, which are practically identical to those obtained when the correction is applied at every interaction step and the solution of IE itself converges.

The Roben principle makes it possible to propose a special method of estimating the accuracy of the computational schemes used to solve IE. Let us assume that geometrical symmetry exists. The eigenfunctions should reflect this symmetry. We can therefore recommend that analysis of the functions $\varphi_n(x)$ be carried out (for sufficiently large n). In the example described above, the values of the functions $\varphi_n(x)$ at the points B and C were practically identical, and their ratio to the values at the centre of the side surface (A) remained constant (to within 4%) for all discretizations, although the functions themselves differed by three orders of magnitude

Discretization	$\varphi_{15}\cdot 10^{3}$	$\varphi_{15}\cdot 10^{7}$	$\varphi^{15}\cdot 10$	$\sigma^{8}\cdot 10$	$\sigma^{15}\cdot 10$
1°	17	(1.15)	7.48 (8.823)	1.9250 (1.9250)	1.9270 (1.9260)
2°	4.3	(1.25)	8.51 (8.847)	2.0670 (2.0670)	2.0680 (2.0680)
3°	1.1	(1.27)	8.76 (8.845)	2.1027 (2.1030)	2.1039 (2.1040)

We know /4/ that the IE of the theory of elasticity for an incompressible medium are identical with the IE for a linearized flow of a viscous incompressible fluid. They have the same spectral properties as the IE of the theory of elasticity (for any value of Poisson's ratio), but in addition they have a resolvent pole at the point $\lambda = 1$ with a single corresponding eigenfunction coinciding with the vector of constant modulus directed along the normal to the boundary surface.

Below, we shall limit ourselves for simplicity to the cases where we have three planes of geometrical and force symmetry, and this practically annihilates the pole of the resolvent at the point $\lambda = -1$. Therefore, it is possible to use the Robens principle. All this enables us to propose a method of estimating the accuracy of the computational schemes for solving the IE of the second fundamental problem. Apart from carrying out the computations at the given value of Poisson's ratio, we must also carry out the computations for an incompressible medium. The deviation of the function $\varphi_n(x)$ from the normal to the surface and the difference in the magnitudes, can serve as the measure in estimating the error.

We shall consider, as an example, the axisymmetric problem of the theory of elasticity in the case when the surface is formed by rotating a square about its diagonal. The surface is not smooth (it has conical apices and a rib), and we therefore have no reason to speak of the convergence of the method of successive approximations and hence of the applicability of the Roben principle. However, the computations carried out under sufficiently fine discretization in the neighbourhood of the non-regular points have shown that over the major part of the surface (except for a small neighbourhood of the irregular points) the deviation of the function φ_n from the normal did not exceed one degree, and the difference in the magnitude did not exceed 3%.

In conclusion we shall turn our attention to the solution of the outer Dirichlet problem for the Laplace equation in the first outer problem of the theory of elasticity. The traditional approach, consisting of representing the required function in the form of the potential of double layer and of a generalized elastic potential of the double layer, leads to IE which have no solutions. Modifications were suggested in /5-9/ to the representations, leading to solvable IE.

Below we give a method of solving the problems in question based on the Roben principle. We require to find the harmonic function $u(p)\,(p \in D^{-})$ satisfying the boundary condition

$$u(q) = f(q),\ q \in \partial D.$$

We solve the corresponding IE by successive approximations and arrive, as a result, at the eigenfunction which we shall denote by $\varphi^{*}(q)$. We further introduce some function $u_1(p)$, harmonic in the region D^{-} and find its trace on ∂D, denoting it by $f_1(q)$. Once again we solve the IE by successive approximations, but using, in this case, the boundary condition $f_1(q)$. We denote the eigenfunction obtained in the iteration process by $\varphi^{**}(q)$. Since the IE has only two linearly independent eigenfunctions, it follows that the ratio $\varphi^{*}(q)/\varphi^{**}(q)$ must be a

constant quantity. We denote this ratio by C.

In the final stage we should consider the boundary value problem for the function $u_2(p)$: $u_2(p) = u(p) - Cu_1(p)$. The process of successive approximations for this function will lead to a convergent algorithm. The solution will be completed by transfer to the function $u(p)$.

In solving the problem of the theory of elasticity, we must begin with the six partial solutions of the boundary value problems and expand the eigenfunction obtained from the initial boundary condition in terms of the functions obtained from the partial solutions.

We note that the method described here was used in solving the second outer problem for an incompressible medium in /10/.

REFERENCES

1. GUNTER N.M., Potential Theory and its Application to the Fundamental Problems of Mathematical Physics. Moscow, Gostekhizdat, 1953.
2. KUPRADZE V.D., Potential Theory Methods in the Theory of Elasticity. Moscow, Fizmatgiz, 1963.
3. PERLIN P.I., Application of the regular representation of singular integrals to the solution of the second fundamental problem of the theory of elasticity. PMM, 40, 2, 1976.
4. ODQWIST F.K.G., Uber die Randwertaufgaben der Hydrodynamik Zäher Flüsigkeiten, Math. Z. 32, 3, 1930.
5. MUSKHELISHVILI N.I., On the solution of fundamental boundary value problem of the theory of Newtonian potential. PMM, 4, 4, 1940.
6. KUPRADZE V.D., On solving the Dirichlet problem for a multiconnected region. Soobshch. Akad. Nauk SSSR, GruzSSR, 1, 8, 1940.
7. SHERMAN D.I., Comment on the Dirichlet problem. Dokl. Akad. Nauk SSSR, 29, 4, 1940.
8. SHERMAN D.I., Spatial problem of the theory of elasticity with specified displacements at the boundary. PMM, 7, 5, 1943.
9. BURCHULADZE T.V., A boundary value problem of the theory of elasticity for multiconnected regions. Tr. Tbil. un-ta, 129, 1968.
10. PERLIN P.I., NOVIKOV A.V. and STUPAK S.F., Application of the potential method to solving the spatial problems of the theory of elasticity for an incompressible fluid. Statika sooruzhenii, Kiev: Kiev. Inzh.-Stroit. In-t, 1978.

Translated by L.K.

PMM U.S.S.R.,Vol.52,No.2,pp.270-274,1988
Printed in Great Britain

0021-8928/88 $10.00+0.00

SOLUTION OF THREE-DIMENSIONAL PROBLEMS OF THE THEORY OF ELASTICITY USING THE MONTE CARLO METHOD*

B.E. POBEDRYA and P.V. CHISTYAKOV

Two versions of the Monte Carlo (MC) method for solving problems of the theory of elasticity are discussed. One uses the process of random walk over spheres to solve the Lamé equations, and the other represents the quantity sought in the form of multiple integrals (e.g. when solving the Cauchy problem for the wave equation of the theory of elasticity in an unbounded space).

The process of random walk over spheres was proposed in /1/ for solving the Laplace equation, and was later used in more complicated problems (an analysis of the work done on this subject can be found in /2, 3/). A solution of the boundary value problem for the Lamé equation was studied for the plane case in /4/, and the possibility of using the MC method for the problems of flexure of plates was discussed in /3, 5/**.(** A further development of methods of solving the problems of plate flexure can be found in the paper by V.M. Ivanov and O.Yu. Kulchitskii. Development and study of effective methods of random walk over circles for solving problems of plate flexure and the plane problem of the theory of elasticity. Deposited at VINITI, No.3270-83, Leningrad, 1983.) Theorems were given in /6/, enabling the initial system of elliptic equations to be replaced by a system of integral equations which

*Prikl.Matem.Mekhan.,52,2,341-345,1988